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On Simultaneous Chebyshev Approximation and Chebyshev Approximation with an Additive Weight

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1. INTRODUCTION

Let $F(x)$ and $f(x)$ be real valued functions, defined on $a \leqslant x \leqslant b$, where $a < b$ are real numbers. Let S denote a nonempty family of real valued functions defined on $[a, b]$. The problem with which we are concerned is that of simultaneously approximating F and f by elements of S. More exactly, we are concerned with the expression

$$\inf_{s \in S} \max\{\sup_{a \leqslant x \leqslant b} |F(x) - s(x)|, \sup_{a \leqslant x \leqslant b} |f(x) - s(x)|\}. \tag{1}$$

Problems involving the expression (1) are discussed in [1, and 2]. If there exists $s^* \in S$ such that the value of (1) actually equals

$$\max\{\sup_{a \leqslant x \leqslant b} |F(x) - s^*(x)|, \sup_{a \leqslant x \leqslant b} |f(x) - s^*(x)|\},$$

then we say that s^* is a best simultaneous approximation to f and F.

For convenience, we use, in what follows, the notation

$$\| g \| = \sup_{a \leqslant x \leqslant b} | g(x)|,$$

for a real valued function $g(x)$ $(a \leqslant x \leqslant b)$.

2. AN EXAMPLE

An interesting special case of the simultaneous approximation problem occurs when F and f are chosen to be continuous functions, and $S = P_n$, the polynomials (with real coefficients) of degree n or less (n a fixed positive integer). One might expect, at first glance, that the element $q \in P_n$ which best approximates the arithmetic mean of F and f is also an element from P_n which best approximates F and f simultaneously, i.e., that, if $q \in P_n$ and

$$\| \tfrac{1}{2}(F + f) - q \| = \inf_{p \in P_n} \| \tfrac{1}{2}(F + f) - p \|,$$

then, also

$$\max\{\| F - q \|, \| f - q \|\} = \inf_{p \in P_n} \max\{\| F - p \|, \| f - p \|\}.$$

If this were actually the case, then the problem of the simultaneous approximation of the two functions F and f would simply be equivalent to the problem of the Chebyshev approximation of a single function, the arithmetic average $(F + f)/2$. However, in general, this is not true, as seen by choosing any F not in P_n, and then choosing $f \in P_n$ such that

$$0 < \| f - \bar{p} \| \leqslant \| F - \bar{p} \|,$$

where $\bar{p} \in P_n$, and $\bar{p}$ is the unique element of P_n which best approximates F, that is

$$\| F - \bar{p} \| = \inf_{p \in P_n} \| F - p \|.$$

(The polynomial f could be chosen, for example, to be $\bar{p} + \| F - \bar{p} \|$). In this example, we have, on the one hand,

$$\left\| \frac{F - \bar{p}}{2} \right\| = \| \tfrac{1}{2}(F + f) - \tfrac{1}{2}(\bar{p} + f) \| = \inf_{p \in P_n} \| \tfrac{1}{2}(F + f) - p \|$$

(i.e., $\frac{1}{2}(\bar{p} + f)$ is the unique element of P_n which best approximates $\frac{1}{2}(F + f)$); and, on the other hand, by the way f was chosen,

$$\| F - \bar{p} \| = \max\{\| F - \bar{p} \|, \| f - \bar{p} \|\} = \inf_{p \in P_n} \max\{\| F - p \|, \| f - p \|\}$$

(i.e., $\bar{p}$ is the unique element of P_n which best approximates F and f simultaneously). Clearly, $\bar{p} \neq \frac{1}{2}(\bar{p} + f)$, since $\| f - \bar{p} \| > 0$.

3. An Equivalence

The following theorem shows, loosely speaking, that the problem of approximating F and f simultaneously is equivalent to what may be described as "the problem of approximating $\frac{1}{2}(F + f)$ with the additive weight function $\frac{1}{2} \mid F - f \mid$".

THEOREM 1. *For a and b real numbers ($a < b$), let $F(x)$ and $f(x)$ denote real valued functions, defined on the interval $a \leqslant x \leqslant b$. Let S denote a non-empty set of real valued functions defined on the interval $a \leqslant x \leqslant b$. Then,*

$$\| \, | \tfrac{1}{2}(F+f) - s \, | + \tfrac{1}{2} \, | F - f | \, \| = \max\{\| F - s \|, \| f - s \|\}$$

for $s \in S$ (and hence

$$\inf_{s \in S} \| \, | \tfrac{1}{2}(F+f) - s \, | + \tfrac{1}{2} \, | F - f | \, \| = \inf_{s \in S} \max\{\| F - s \|, \| f - s \|\}).$$

Proof. The proof is based on the following lemma.

LEMMA 1. *If m and n are real numbers, then*

$$| \tfrac{1}{2}(m+n) | + \tfrac{1}{2} \, | m - n \, | = \max\{| m |, | n |\}.$$

Proof. In the first place, the equation to be proved remains unaltered, either when m is replaced by $-m$, or when n is replaced by $-n$, or both at once; hence it may be assumed that both $m \geqslant 0$ and $n \geqslant 0$. In the second place, the equation to be proved remains unaltered when m and n are interchanged; hence it may be assumed that $m \geqslant n$. Therefore, it may be assumed, without loss, that $m \geqslant n \geqslant 0$. But then the desired equality is obvious.

Returning to the proof of the theorem, we identify $F(x) - s(x)$ with m, and $f(x) - s(x)$ with n, in the lemma, to obtain

$$| \tfrac{1}{2}(F(x) + f(x)) - s(x) | + \tfrac{1}{2} \, | F(x) - f(x) | = \max\{| F(x) - s(x) |, f(x) - s(x) |\},$$

on $a \leqslant x \leqslant b$. Then, taking the supremum, on both sides of this last equation, over $a \leqslant x \leqslant b$, and noticing that

$$\sup_{a \leqslant x \leqslant b} \max\{| F(x) - s(x) |, | f(x) - s(x) |\} = \max\{\| F - s \|, \| f - s \|\},$$

yields directly the desired conclusion of Theorem 1.

4. REFORMULATION OF THE EQUIVALENCE

Theorem 1 can be looked at from a different point of view. That is, the problem of approximating, with an additive non-negative weight, a given function, by elements of S, is equivalent to simultaneously approximating two appropriate functions F and f by elements of S. We state the result formally.

THEOREM 2. *For a and b real numbers ($a < b$), let $g(x)$ and $h(x)$ denote* *real valued functions on the interval $a \leqslant x \leqslant b$. Let S denote a nonempty set of real valued functions defined on the interval $a \leqslant x \leqslant b$. Then, for every $s \in S$ the following relation holds*:

$$\| \, | h - s | + | g | \, \| = \max\{\|(h + g) - s\|, \|(h - g) - s\|\}$$

(*and hence*

$$\inf_{s \in S} \| \, | h - s | + | g | \, \| = \inf_{s \in S} \max\{\|(h + g) - s\|, \|(h - g) - s\|\}).$$

Proof. The proof follows from Theorem 1, by identifying $h + g$ with F, and $h - g$ with f.

5. A Question

The problem of the simultaneous approximation of complex valued functions has been considered in [3]. The following question arises naturally: Is an equivalence result, such as proved above for *real* valued functions, also valid for *complex* valued functions? If there is, the method of proof cannot be identical to that in the real case considered above, because the crucial Lemma 1 does not carry over to complex numbers, as can be readily seen by taking $m = 1 + i$ and $n = 1 - i$.

References

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